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Computing Self-Intersections of Closed Geodesics on Finite-Sheeted Covers of the Modular Surface

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Abstract. An algorithm is given for deciding whether a closed geodesic on a finite-sheeted cover of the modular surface has self-intersections; if it does, the algorithm gives them in the order they occur in traversing the geodesic. The following general result on geodesics is proved: any closed geodesic on a Riemann surface R can be lifted to a simple closed geodesic on some finite-sheeted cover of R. In the last two sections the connection with the stabilizer (under the modular group) of a Markov quadratic irrationality is discussed.

0. Introduction. The main object of this paper is to give an algorithm for deciding whether a closed geodesic on a finite-sheeted cover of the modular surface has self-intersections, and if it does, to give them in the order they occur in traversing the geodesic. Birman and Series [3], [4] have a very different approach to this problem, which has yielded nice insights for many Riemann surfaces.

Let $\Gamma(1) = \mathrm{PSL}(2, \mathbf{Z})$ be the modular group and $S_1 = H^+/\Gamma(1)$ the modular surface, where H^+ is the upper half-plane. If $\Gamma \subset \Gamma(1)$ is of finite index, then $S = H^+/\Gamma$ is a finite-sheeted cover of S_1, and conversely. Let $\sigma \in \Gamma(1)$ be a primitive hyperbolic transformation and A_σ its axis. $A_\sigma/\Gamma(1)$ is a closed geodesic on S_1; conversely, every such geodesic lifts to a conjugacy class of axes of primitive hyperbolics in $\Gamma(1)$. But because Γ is of finite index, σ^k is in Γ for some k. Hence A_σ projects to a closed geodesic on S as well, and this geodesic covers $A_\sigma/\Gamma(1)$ on S_1. Every closed geodesic arises in this way.

Theorem 2.1, which led to our algorithm, provides a relationship between these projections:

THEOREM 2.1. *Let L be a closed geodesic on a Riemann surface R. Then there is a finite-sheeted cover S of R with the property that L lifts to a* ***simple*** *closed geodesic on S.*

It should be noted that there are no nontrivial simple closed geodesics on S_1, so the theorem is never vacuous in that case. For the surface S_1 and with S a principal congruence subgroup, we had an explicit proof of this result. Morris Newman [7] provided a key algebraic result that made the generalization possible. The proof is in Section 2.

We arrived at the algorithm while searching for simple closed geodesics (scg) on the surface $H^+/\Gamma(3)$. This is a sphere with 4 punctures and there are infinitely many

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homotopy classes, each containing a unique scg. The index of $\Gamma(3)$ in $\Gamma(1)$ is 12, so the fundamental region (FR) for $\Gamma(3)$ consists of 12 copies of the FR for $\Gamma(1)$. But in the algorithm all the geometry takes place in the FR of $\Gamma(1)$, with its 3 sides at great remove from the real axis. All "irrational" calculation pertains only to this FR. All other calculations involve only integer arithmetic. We believe this is a distinct advantage—one that can be exploited whenever we have a specific subgroup of finite index. The algorithm appears in Section 1.

In our work on $\Gamma(3)$ we are given the endpoints of a hyperbolic axis, i.e., conjugate quadratic irrationals α, $\bar{\alpha}$, but what we need is the primitive hyperbolic transformation of $\Gamma(1)$ fixing α and $\bar{\alpha}$. This calculation, involving the smallest positive solution of a certain Pell equation, is probably well-known, but we could not find it in the literature. It is the subject of Section 3.

To describe the last section, we must define the Markov spectrum (MS) and Markov quadratic irrationalities (MQI). Hurwitz proved that for every irrational α the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{\sqrt{5}\,q^2} \tag{0.1}$$

is satisfied by infinitely many reduced fractions and that $\mu_1 = \sqrt{5}$ is best possible. The value $\sqrt{5}$ is attained for $\alpha \sim \lambda_1 = (1 + \sqrt{5})/2$, the equivalence being under the extended modular group (determinant ± 1). For $\alpha \nsim \lambda_1$ we can get (0.1) with μ_1 replaced by $\mu_2 = \sqrt{8}$ and this is attained for $\alpha \sim \lambda_2 = 1 + \sqrt{2}$. The numbers μ_i increase monotonically to 3; they constitute the MS.

The λ_i are the MQI. They can be described as follows. Consider the Diophantine equation

$$x^2 + y^2 + z^2 = 3xyz, \qquad 1 \leqslant x \leqslant y \leqslant z. \tag{0.2}$$

Ordering its solutions by the size of z, we have

$$\mu_i = \sqrt{9 - \frac{4}{z_i^2}}, \qquad \lambda_i = \frac{1}{2} + \frac{y_i}{x_i z_i} + \frac{1}{2}\sqrt{9 - \frac{4}{z_i^2}}. \tag{0.3}$$

(Here we assume the well-known conjecture that z_i determines x_i and y_i uniquely.) See Koksma [6, Chapter 3] for all of this.

Halting the algorithm for a given hyperbolic $\sigma \in \Gamma(1)$ depends on finding the least positive $k_\sigma = k$ for which $\sigma^k \in \Gamma$. When $\Gamma = \Gamma(3)$ we have $\Gamma(1)/\Gamma(3) \simeq A_4$, the alternating group on 4 letters. Since A_4 has no elements of order 4, 6, or 12, we have $k = 1$, 2, or 3 [8, p. 31]. Our computations, however, suggested that for σ fixing λ_i, an MQI, $k = 2$. In Section 4 we prove that, for such σ, $k \leqslant 2$, and that proving $k = 2$ depends on a congruence property of the smallest solution of the corresponding Pell equation.

Lastly, we mention that using our algorithm on various λ_i suggested the conjecture:

(0.4) A_σ projects to a *simple* closed geodesic on $H^+/\Gamma(3)$ if and only if σ fixes a MQI.

Using methods of number theory, topology, and differential geometry we were subsequently able to prove this conjecture. (See [2].) Combined with an asymptotic formula for the number of $z_i \leqslant X$ given by Zagier [10], (0.4) gives upper and lower bounds for the number of scg on $H^+/\Gamma(3)$ whose hyperbolic length is $\leqslant X$.

1. An Algorithm for Finding Self-Intersections of a Closed Geodesic on a Modular Surface of Finite Index in $\Gamma(1)$. We shall assume that the closed geodesic is given as the axis of an explicit hyperbolic transformation A. We can also assume the axis intersects the standard fundamental region (SFR) for $\Gamma(1)$, entering at a point z_0 that lies on a vertical side of the SFR. This can be achieved by an effective conjugation of A. Thus we have fixed points of A, $\xi_A < \xi'_A$, that satisfy $\xi'_A - \xi_A > 1$.

Since our surface is of finite index, there is a least positive integer $n_A = n$ such that $A^n \in \Gamma$. Our implementations of this algorithm test each integer $n = 1, 2, 3, \ldots$ to see if $A^n \in \Gamma$. For many subgroups of $\Gamma(1)$ there is a very simple membership test (Rankin [8, pp. 63–65]).

We now have determined A, $n = n_A$, $\xi = \xi_A$, and $\xi' = \xi'_A$. We are going to follow the axis from z_0 towards $A^n(z_0)$. As we first pass out of the SFR at z_1, say, we determine which of $S = (1, 1 : 0, 1)$, S^{-1}, or $T = (0, -1 : 1, 0)$ returns the path (from z_1 towards $A^n(z_0)$) to the SFR. Call this L_1. Apply L_1 to the axis of A; the image enters the SFR at $L_1(z_1)$ and moves towards $L_1 \circ A^n(z_0)$. Again we leave the SFR at z_2 and map the segment proceeding from z_2 towards $L_1 \circ A^n(z_0)$ back to the SFR by L_2, one of S, S^{-1}, or T. Continuing in this way, we obtain a sequence $\{L_j\}$, $j = 1, \ldots, k = k(A^n)$. Furthermore, $A^n = L_k \circ L_{k-1} \circ \cdots \circ L_2 \circ L_1$, since the segments emanating from z_0 and $A^n(z_0)$ are Γ-equivalent and a fortiori $\Gamma(1)$-equivalent. Because of the relation $(TS)^3 = I$, there are many representations of A^n as a word in the generators S and T, but the word obtained as above is about as short as possible (Beardon [1, Theorem 5]).

Our first task is to determine the L's. We assume we are proceeding from z_0 to ξ, the other case being no different. Find m such that $m - \frac{1}{2} < \xi < m + \frac{1}{2}$. By our normalization, $m \leqslant 0$. Proceeding from z_0 towards ξ, we have $|m|$ or $|m| - 1$ instances of S to begin our sequence of L's, according as we are in case (a) or (b) in Figure 1:

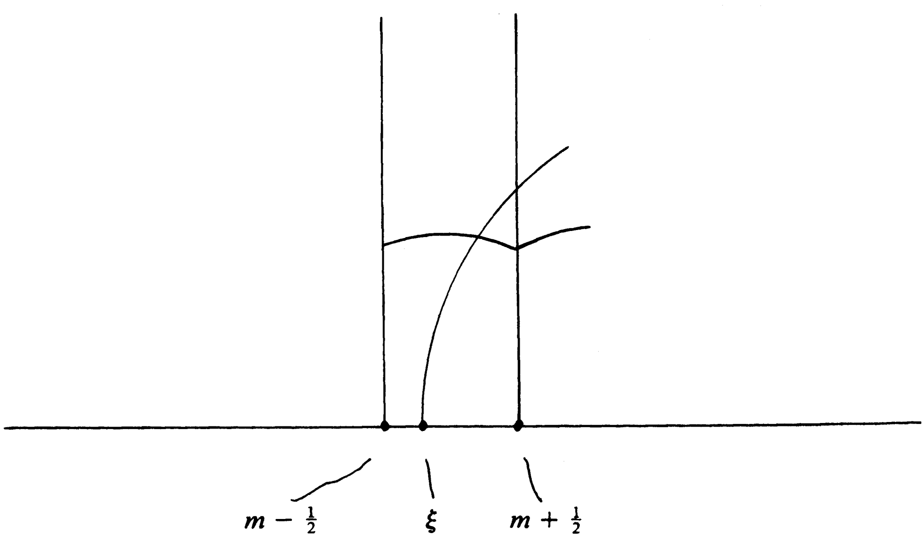

FIGURE 1 (a)

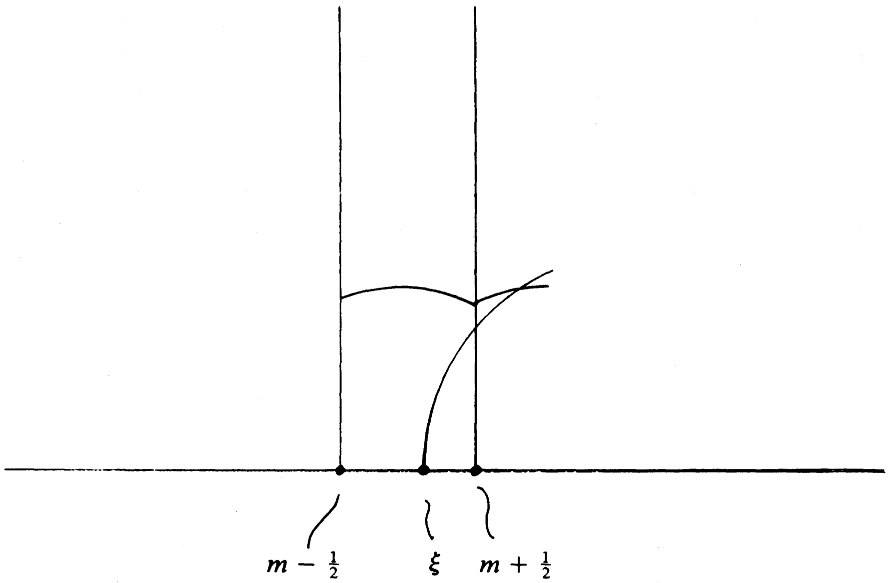

FIGURE 1 (b)

Determining the intersection of the axis with $x = m + \frac{1}{2}$ (solving a quadratic equation) distinguishes the cases. We include the limiting case in which the intersection occurs at height $\sqrt{3}/2$ in case (b).

After this initial string of S's, the next L is of course T. Calculate $T \circ S^{|m|}(\xi) = \xi_1$ and $T \circ S^{|m|}(\xi') = \xi_1'$, and determine whether or not $\xi_1 < \xi_1'$. We know we are continuing from z_1 towards ξ_1. Now just as with our analysis from z_0, we obtain a sequence of transformations S or S^{-1} according as $\xi_1 < \xi_1'$ or not. The number of L's in this sequence is $|m_1|$ or $|m_1| - 1$, where $m_1 - \frac{1}{2} < \xi_1 < m_1 + \frac{1}{2}$, as before. The next L is T.

We continue in this way until $\xi_t = \xi$ and $\xi_t' = \xi'$ in this order. (Note that if $\xi_t = \xi'$, $\xi_t' = \xi$ occurs, it means that the axis has fixed points of order 2 on it.) Up to this point we have determined $L_1, \ldots, L_t$ and we know that $A = L_t \circ \cdots \circ L_1$, since the product fixes ξ, ξ' and is primitive because $\xi_t = \xi$, $\xi_t' = \xi'$ does not occur earlier. The full sequence of L's is just $L_1, \ldots, L_t$ repeated n-times.

Next, define $Q_k = L_k \circ \cdots \circ L_1$. We see that Q_k maps a segment of the axis between z_0 and $A(z_0)$ into the SFR. These Q_k are the only elements of $\Gamma(1)$ with this property. This means that if β and β' lie in the segment of the axis between z_0 and $A(z_0)$ with β nearer z_0, and β is $\Gamma(1)$-equivalent to β', then $Q_k(\beta) = Q_j(\beta')$ for some $j > k$. Thus $\beta = Q_k^{-1} \circ Q_j(\beta')$.

Now we compute all $Q_k^{-1} \circ Q_j$, $nt \geqslant j > k$, and check for $\Gamma(1)$-self-intersections. This involves computing $\xi_{k,j} := Q_k^{-1} \circ Q_j(\xi)$ and $\xi_{k,j}' := Q_k^{-1} \circ Q_j(\xi')$ and checking for the following 4 incidence patterns (recall $\xi < \xi'$):

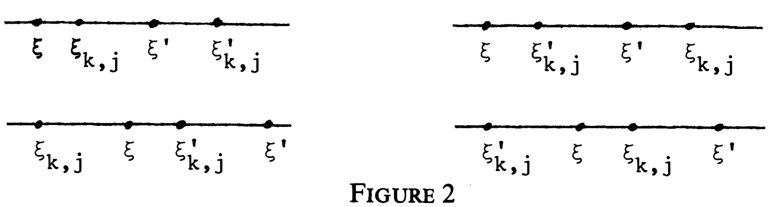

FIGURE 2

If a self-intersection occurs, compute it to make sure it lies in $[z_0, A^n z_0]$—this avoids finding the same intersection more than once. We now have a set of self-intersections ordered by the distance of β from z_0. This sequence actually gives all $\Gamma(1)$-intersections in the order they occur in traversing the axis from z_0 to $A^n(z_0)$, which is n-times around the curve $A/\Gamma(1)$.

Finally, to find Γ-intersections, simply check which of the $\Gamma(1)$-intersections have $Q_k^{-1} \circ Q_j \in \Gamma$. The efficacy of this algorithm lies in the fact that the geometry is carried out on the SFR of $\Gamma(1)$ with its 2 simple identifications and 4 long sides whose 3 vertices are far from the real axis. Γ enters only in that one needs to check membership therein—an integer arithmetic problem for many congruence subgroups. This is much preferable to dealing with a fundamental region with many short sides and closely-spaced vertices near the real axis, as the algorithm of Poincaré requires [9, pp. 192–194].

2. Proof of the Lifting Theorem. As we have said, this theorem relies on a result of M. Newman.

THEOREM 2.2. *Let Γ be a Fuchsian group, $\{\alpha_1, \ldots, \alpha_k\}$ a finite subset of Γ not containing* 1, $\gamma \in \Gamma$ *such that* $\gamma\alpha_i \neq \alpha_i\gamma$, $1 \leqslant i \leqslant k$. *Then there is a subgroup Γ_1 of Γ such that* $(\Gamma : \Gamma_1) < \infty$, $\gamma \in \Gamma_1$, $\alpha_i \notin \Gamma_1$, $i = 1, \ldots, k$.

Proof of Theorem 2.1. Let the Riemann surface $R = H^+/\Gamma$, where Γ is a Fuchsian group, and let π: $H^+ \to R$ be the projection map. Let L be a closed geodesic in R. We observe that L can have only finitely many self-intersections, since a geodesic on a surface of constant negative curvature is a real analytic curve. Now L lifts to a countable set of axes of primitive hyperbolic elements in Γ. Let γ be one such with axis A_γ.

Choose a point a on A_γ such that $\pi(a)$ is not a self-intersection of L, and let $b = \gamma(a)$. As we move from a to b along A_γ, the projection traverses L once. Also, moving from a to b we pass through finitely many fundamental regions of Γ. Hence there are at most finitely many $1 \neq \alpha_i \in \Gamma$ with $\alpha_i(z) = z'$; $z, z' \in [a, b)$. Note that $\alpha_i\gamma \neq \gamma\alpha_i$, as else α_i and γ would have the same fixed points, implying that $\alpha_i = \gamma^m$, $m \neq 0$. This is not possible, as $z, z' \in [a, b)$.

Now apply Theorem 2.2, obtaining a finite-index subgroup Γ_1 of Γ that contains γ but excludes $\{\alpha_i\}$. Clearly γ is primitive also in Γ_1. On the surface $S := H^+/\Gamma_1$, the axis A_γ projects to a closed geodesic L_1 and L_1 is described once as A_γ is traversed from a to b.

We claim that A_γ/Γ_1 is simple. If not, there is a $\sigma \in \Gamma_1$ with

$$A_\gamma \neq \sigma(A_\gamma) \cap A_\gamma \neq \varnothing. \tag{2.1}$$

In particular, $\sigma \neq 1$. Let $\sigma(A_\gamma) \cap A_\gamma = z_1 \in H^+$. Without loss of generality, $z_1 \in [a, b)$.

Let $z_1 = \sigma(z_2)$ with $z_2 \in A_\gamma$. There exist integers m, n such that $\gamma^m(z_1)$, $\gamma^n(z_2) = \gamma^n \circ \sigma^{-1}(z_1) \in [a, b)$. Since they are Γ_1-equivalent and both lie in $[a, b)$, Theorem 2.2 forces $\gamma^n \circ \sigma^{-1} = \gamma^m$, or

$$\gamma^{n-m} = \sigma. \tag{2.2}$$

Whether $n = m$ or $n \neq m$, this contradicts (2.1). Finally, since $\Gamma_1 \subset \Gamma$, A_γ/Γ_1 covers A_γ/Γ.

3. Constructing a Generator for the $\Gamma(1)$-Stabilizer of a Quadratic Irrationality.

THEOREM 3.1. *Let α be a quadratic irrationality whose primitive polynomial is*

$$a_2 z^2 + a_1 z + a_0 = 0, \qquad a_2 \neq 0, \qquad (a_2, a_1, a_0) = 1. \tag{3.1}$$

Then the $\Gamma(1)$-stabilizer of α is generated by

$$\begin{pmatrix} \dfrac{L - Ka_1}{2} & -Ka_0 \\ Ka_2 & \dfrac{L + Ka_1}{2} \end{pmatrix}, \tag{3.2}$$

where (L, K) is the positive solution of the Pell equation

$$K^2(a_1^2 - 4a_0 a_2) + 4 = L^2$$

with minimal K (and thus minimal L).

Remark. In what follows, we are regarding $\Gamma(1)$ as a transformation group, i.e., we identify $V \in \Gamma(1)$ with $-V$.

Proof. Every quadratic polynomial with roots α, $\bar{\alpha}$ is of the form

$$ua_2 x^2 + ua_1 x + ua_0 = 0, \qquad u \neq 0,$$

and conversely.

Now suppose $V = (a, b\colon c, d) \in \Gamma(1)$ has fixed points α, $\bar{\alpha}$. Then

$$c\alpha^2 + (d - a)\alpha - b = 0.$$

Hence,

$$c = Ka_2, \quad d - a = Ka_1, \quad -b = Ka_0, \quad K \neq 0, \tag{3.3}$$

where clearly K is rational. Let $K = K_1/K_2$, $(K_1, K_2) = 1$. We have $K_2 c = K_1 a_2$, yielding $K_2 \mid a_2$. Similarly, $K_2 \mid a_1$, $K_2 \mid a_0$. Since $(a_2, a_1, a_0) = 1$, we must have $K_2 = 1$, i.e., $K \in \mathbf{Z}$. Hence, every $V = (a, b\colon c, d) \in \Gamma(1)$ with fixed points α, $\bar{\alpha}$ satisfies (3.3) with an integer K. Furthermore, there exists an integer L such that

$$K^2(a_1^2 - 4a_0 a_2) + 4 = L^2. \tag{3.4}$$

The last statement is proved as follows. We have

$$(a + d)^2 - (a - d)^2 = 4 + 4bc = 4 - 4K^2 a_0 a_2,$$
$$(a + d)^2 = 4 + K^2(a_1^2 - 4a_0 a_2) = L^2$$

for an integer L, since $a + d \in \mathbf{Z}$. Note that Ka_1 and L have the same parity.

Conversely, if $a, b, c, d \in \mathbf{Z}$ satisfy (3.3) and (3.4) for some integers K, L, then $V = (a, b\colon c, d) \in \Gamma(1)$ and has fixed points α, $\bar{\alpha}$. This is easily checked. So we have

LEMMA 3.2. *The subgroup Γ_α consists of the $V \in \Gamma$ satisfying* (3.1) *and* (3.2). *Here we have written Γ for $\Gamma(1)$.*

An element of Γ_α is therefore of the form (3.2), where we may assume $K, L > 0$. Let L_1 be the smallest solution of (3.4) and write K_1 for the corresponding K. Let

$$V_1 = \begin{pmatrix} (L_1 - K_1 a_1)/2 & -K_1 a_0 \\ K a_2 & (L_1 + K_1 a_1)/2 \end{pmatrix}.$$

We shall show that V_1 generates Γ_α.

LEMMA 3.3. *Let* $A = (\alpha, \beta\colon \gamma, \delta) \in \mathrm{SL}(2, \mathbf{R})$ *be hyperbolic. Then,*

$$A^n = u_n A + v_n I, \quad v_{n+1} = -u_n, \qquad n \geqslant 0; \quad u_0 = 0, \quad v_0 = 1, \tag{3.5}$$
$$u_n = 2^{n-1} \sinh n\theta / \sinh\theta, \qquad n \geqslant 1, \theta > 0,$$

where θ *is defined by*

$$\alpha + \delta = \pm 2\cosh\theta.$$

The proof is by induction on n, using $A^2 = (\alpha + \delta)A - I$.

Proceeding with the proof of the theorem, we let W be a generator of the cyclic group Γ_α, where, as noted above, we regard Γ as a transformation group. Then

$$W = \begin{pmatrix} (L_0 - K_0 a_1)/2 & -K_0 a_0 \\ K_0 a_2 & (L_0 + K_0 a_1)/2 \end{pmatrix}$$

for a solution (K_0, L_0) of (3.4). Suppose $V_1 = W^n$, $n \geqslant 1$. Then, by Lemmas 3.2 and 3.3,

$$|K_1 a_2| = |u_n K_0 a_2| = 2^{n-1} \frac{\sinh n\theta}{\sinh\theta} |K_0 a_2|, \qquad |K_1| \geqslant |K_0|.$$

By hypothesis $|K_1| \leqslant |K_0|$, hence $|K_1| = |K_0|$, which implies $n = 1$. That is, $V_1 = W$, a generator. This concludes the proof of Theorem 3.1.

4. The Exponent of a Markov Quadratic Irrationality. A Markov quadratic irrationality (MQI) is a number of the form

$$\alpha = \frac{1}{2} + \frac{y}{xz} + \frac{1}{2}\sqrt{9 - \frac{4}{z^2}}, \tag{4.1}$$

where x, y, z $(x \leqslant y \leqslant z)$ is a Markov triple. Writing out the equation satisfied by α and cancelling common factors in the coefficients, we find, by comparison with (3.1), that

$$a_2 = \frac{x^2 z}{(z,2)}, \quad a_1 = \frac{x(xz + 2y)}{(z,2)}, \quad a_0 = \frac{-2x^2 z + 4xy - z}{(z,2)}. \tag{4.2}$$

Here we have used the coprimality of x, y, z in pairs (Cassels [5, p. 28]).

Now (4.2) and Theorem 3.1 give

THEOREM 4.1. *With α as in* (4.1) *we have that the primitive matrix B_α generating the stabilizer of α in $\Gamma(1)$ is*

$$\begin{pmatrix} \dfrac{L + x(2y+xz)K}{2d} & K(2x^2z - 4xy + z)d^{-1} \\ x^2zKd^{-1} & \dfrac{L - x(2y+xz)K}{2d} \end{pmatrix}, \tag{4.3}$$

where $d = (z, 2)$. L and K are defined as in Theorem 3.1; *explicitly K (and therefore L) is the minimal positive solution of the Pell equation*

$$x^4(9z^2-4)d^{-2}K^2 + 4 = L^2. \tag{4.4}$$

Moreover, the smallest power of B_α lying in $\Gamma(3)$ is 1 *or* 2 *according as* $(3, K) = 3$ *or* 1.

Proof. Only the last statement needs proof. We easily deduce from the Markov equation that $3 \nmid xz$. Equation (4.4) now forces $3 \mid KL$, $K \not\equiv L$ (3). Assume $3 \mid L$, then $L \mid Ld^{-1} = \operatorname{trace} B_\alpha$. Hence,

$$B_\alpha^2 \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \pmod 3,$$

i.e., $B_\alpha^2 \in \Gamma(3)$. Next, let $3 \nmid L$, then $3 \mid K$. It follows that

$$B_\alpha \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \pmod 3,$$

i.e., $B_\alpha \in \Gamma(3)$. Finally, $3 \nmid K$ implies $B_\alpha \notin \Gamma(3)$ since $x^2zKd^{-1} \not\equiv 0$ (3). This completes the proof.

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